# Hall and Heat Transfer Effects on the Steady Flow of a Sisko Fluid

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This investigation is concerned with the flow and heat transfer analysis between two disks rotating about non-coaxial axes normal to the disks. The constitutive equation of an incompressible Sisko fluid is used. The fluid is electrically conducting and the Hall effect is taken into account. Analytic solutions of the governing nonlinear problem is obtained by homotopy analysis method (HAM). The graphs are presented and discussed. Finally a comparison is made between the results of viscous and Sisko fluids.

*Key words:* Sisko Fluid; Nonlinear Problem; HAM Solution; Heat Transfer.

## 1. Introduction

Nature is abundant of fluids for which the Navier-Stokes equations are inadequate. Such fluids include shampoo, blood, mud, ice cream, clay, coating, paints, ketchup, certain oils and greases, polymer melts, many emulsions etc. Because of several industrial technological applications, the non-Newtonian fluids are considered more important than viscous fluids. Unlike the viscous fluids there is not a single constitutive equation available in the literature by which the behaviour of all the non-Newtonian fluids can be analyzed. In fact this is due to the diversity of non-Newtonian fluids in nature. The constitutive equations of non-Newtonian fluids involve rheological parameters. Except in the case of some basic flows, the constitutive equations of non-Newtonian fluids give rise to more complexities in the momentum equation. The resulting equations are of higher order than the Navier-Stokes equations and the adherence boundary conditions are insufficient for the determinacy [1]. The equations of non-Newtonian fluids are much complicated and making the task of obtaining the accurate solutions is a difficult one. Moreover the magnetohydrodynamic (MHD) features of non-Newtonian fluids add further complications in the governing equations. Such flow of an electrically conducting fluid under the action of a constant magnetic field has applications in many devices such as MHD power generators, MHD pumps and accelerators etc. Examples include flow of nuclear fuel slurries, flow of liquid metals and alloys, flow of plasma, flow of mercury amalgams, lubrications of heavy oils and greases. In spite of all these challenges various workers [2 – 14] in that field are engaged recently in obtaining the analytic solutions of non-Newtonian fluids.

It is known that Berker [15] discussed the possibility of the exact solution for viscous flow caused by the non-coaxial rotation of a disk and fluid at infinity. He analyzed the flow between two rotating disks of some angular velocity. It is proved that an infinite number of solutions exists for this flow configuration. An extra condition is necessary for a unique solution. However there is a unique solution when there is a single disk. Coirier [16] examined the flow due to non-coaxial rotation of a disk with different angular velocities and fluid at infinity. The steady flow engendered by non-coaxial rotation of a porous disk and viscous fluid at infinity has been presented by Erdogan [17, 18]. Also the asymptotic solutions for uniform suction and blowing at the disk are obtained in [17, 18]. The flow analysis of [17] is extended to the MHD and heat transfer situations by Murthy and Ram [19]. The unsteady flows created by the non-coaxial rotations of the disks and viscous hydrodynamic and MHD fluids in various situations have been studied by Pop [20], Kasiviswanathan and Rao [21], Erdogan [22 – 24], and Hayat et al. [25 – 29]. A large number of relevant studies for flows of Newtonian and non-Newtonian fluids between parallel disks rotating about a common axis have been presented in a review article by Rajagopal [30]. Recently Ersoy [31] discussed the flow due to a pull with constant velocity of eccentric rotating disks with the same angular velocity.

In this attempt we discuss the Hall and heat transfer effects on the steady flow of an incompressible and electrically conducting Sisko fluid. The fluid is between the insulated disks of constant temperatures. The flow is caused by a sudden pull of eccentric rotating disks. The governing nonlinear problem is solved by a powerful analytic technique namely the homotopy analysis method (HAM) [32–44]. It is only recently that successful attempts have been made to compute analytically the flows of viscoelastic fluids employing HAM. The obtained series solutions of velocity and temperature are sketched and analyzed in detail.

## 2. Mathematical Formulation

Consider the electrically conducting Sisko fluid between two infinite disks rotating with angular velocity $\boldsymbol{\Omega}$ about two non-coaxial axes with distant $2l$. The $z$-axis is chosen normal to the disk. The fluid is electrically conducting in the presence of a constant applied magnetic field. The disks at $z=h$ and $z=-h$ are pulled with constant velocities $\mathbf{U}$ and $-\mathbf{U}$, respectively.

The appropriate boundary conditions are [31]:

$$\begin{aligned} &u=-\Omega(y-l)+U_1, \quad v=\Omega x+U_2,\\ &w=0 \ \text{at} \ z=h,\\ &u=-\Omega(y+l)-U_1, \quad v=\Omega x-U_2,\\ &w=0 \ \text{at} \ z=-h. \end{aligned} \tag{1}$$

The above boundary conditions suggest the velocity of the form [20, 22–24]

$$u=-\Omega y+f(z), \quad v=\Omega x+g(z), \quad w=0. \tag{2}$$

The equations governing the MHD steady flow of an incompressible fluid are

$$\operatorname{div}\mathbf{V}=0, \tag{3}$$

$$\rho(\mathbf{V}\cdot\nabla)\mathbf{V}=\operatorname{div}\mathbf{T}+\mathbf{J}\times\mathbf{B}, \tag{4}$$

$$\nabla\cdot\mathbf{B}=0, \quad \nabla\times\mathbf{B}=\mu_m\mathbf{J}, \quad \nabla\times\mathbf{E}=0. \tag{5}$$

In the above equations $\mathbf{V}$, $\rho$, $\mathbf{J}$, $\mu_m$, $\mathbf{E}$, and $\sigma$ are velocity, fluid density, current density, magnetic permeability, total electric field, and electrical conductivity, respectively. Furthermore, the generalized Ohm's law in the presence of a Hall current is

$$\mathbf{J}+\frac{\omega_e\tau_e}{B_0}(\mathbf{J}\times\mathbf{B}_0)=\sigma\left[\mathbf{E}+\mathbf{V}\times\mathbf{B}+\frac{1}{en_e}\nabla p_e\right], \tag{6}$$

where $e$ is the electron charge, $\mathbf{B}_0$ is the applied magnetic field, $\omega_e$ is the cyclotron frequency of electrons, $\tau_e$ is the electron collision time, $n_e$ is the number density of the electron and $p_e$ is the electron pressure. Here ion-slip and thermo electric effects are neglected. The induced magnetic field is negligible. Also $\omega_e\tau_e \approx O(1)$ and $\omega_i\tau_i \ll 1$, where $\omega_i$ and $\tau_i$ are the cyclotron frequency and collision time for ions, respectively.

The Cauchy stress tensor for a Sisko fluid is [45]

$$\mathbf{T}=-p\mathbf{I}+\mathbf{S}, \tag{7}$$

$$\begin{aligned} &\mathbf{S}=\left[a+b\left|\frac{1}{2}\operatorname{tr}(\mathbf{A}_1^2)\right|^{\frac{n-1}{2}}\right]\mathbf{A}_1,\\ &\mathbf{A}_1=\mathbf{L}+\mathbf{L}^T, \quad \mathbf{L}=\operatorname{grad}\mathbf{V}. \end{aligned} \tag{8}$$

It should be noted that for $b=0$ and $a=\mu$, (8) reduces to the equation of a viscous fluid. Invoking (2) into (8) one obtains

$$\begin{aligned} &S_{xx}=S_{yy}=S_{zz}=S_{xy}=S_{yx}=0,\\ &S_{xz}=S_{zx}=\left[a+b\left(\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2+\left(\frac{\mathrm{d}g}{\mathrm{d}z}\right)^2\right)^{\frac{n-1}{2}}\right]\frac{\mathrm{d}f}{\mathrm{d}z},\\ &S_{yz}=S_{zy}=\left[a+b\left(\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2+\left(\frac{\mathrm{d}g}{\mathrm{d}z}\right)^2\right)^{\frac{n-1}{2}}\right]\frac{\mathrm{d}g}{\mathrm{d}z}. \end{aligned} \tag{9}$$

Through (2), (3) is automatically satisfied and (4) along with (6) yields

$$\begin{aligned} \frac{\partial p}{\partial x}=&\rho\Omega[\Omega x+g(z)]+\frac{\partial S_{xz}}{\partial z}\\ &+\frac{\sigma B_0^2(1+\mathrm{i}\phi)}{1+\phi^2}\left(\frac{Q}{2h}-f(z)\right), \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{\partial p}{\partial y}=&\rho\Omega[\Omega y-f(z)]+\frac{\partial S_{yz}}{\partial z}\\ &+\frac{\sigma B_0^2(1+\mathrm{i}\phi)}{1+\phi^2}\left(\frac{P}{2h}-g(z)\right), \end{aligned} \tag{11}$$

$$\frac{\partial p}{\partial z}=0, \tag{12}$$

when

$$Q=\int_{-h}^{h}f(z)\mathrm{d}z, \quad P=\int_{-h}^{h}g(z)\mathrm{d}z, \tag{13}$$

$p\neq p(z)$ and $\phi=\omega_e\tau_e$ is the Hall parameter.

The relevant boundary conditions are of the form

$$f(z)=\Omega l+U_1,\quad g(z)=U_2 \text{ at } z=h, \tag{14}$$

$$f(z)=-\Omega l-U_1,\quad g(z)=-U_2 \text{ at } z=-h. \tag{15}$$

From (10) – (13) we may obtain

$$\Omega\rho g(z)+\frac{\partial S_{xz}}{\partial z}-Hf(z)=C_1, \tag{16}$$

$$-\Omega\rho f(z)+\frac{\partial S_{yz}}{\partial z}-Hg(z)=C_2, \tag{17}$$

where

$$H=\frac{\sigma B_0^2(1+\mathrm{i}\phi)}{1+\phi^2}. \tag{18}$$

Equations (10) and (11) upon integration give

$$p=p_0+\frac{1}{2}(x^2+y^2)+\left[C_1+\frac{HQ}{2h}\right]x+\left[C_2+\frac{HP}{2h}\right]y,$$

where $p_0$ is the reference pressure and $C_i$ $(i=1,2)$ are the arbitrary constants. In above equation there arises a pressure gradient between the two disks that corresponds to the Poiseuille flow when $C_1+\frac{HQ}{2h}\neq 0$ and $C_2+\frac{HP}{2h}\neq 0$. In absence of the Poiseuille flow and to ensure the symmetry of the velocity distribution about the disk $z=0$, we choose

$$C_1=-\frac{HQ}{2h},\quad C_2=-\frac{HP}{2h}. \tag{19}$$

Substituting (9) and (18) into (16) and (17) we have

$$a\frac{\mathrm{d}^2F}{\mathrm{d}z^2}+b\left[\frac{\mathrm{d}^2F}{\mathrm{d}z^2}\left(\frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}\bar F}{\mathrm{d}z}\right)^{\frac{n-1}{2}}+\frac{n-1}{2}\frac{\mathrm{d}F}{\mathrm{d}z}\left(\frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}\bar F}{\mathrm{d}z}\right)^{\frac{n-3}{2}}\left(\frac{\mathrm{d}^2F}{\mathrm{d}z^2}\frac{\mathrm{d}\bar F}{\mathrm{d}z}+\frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}^2\bar F}{\mathrm{d}z^2}\right)\right]-\left(\frac{\sigma B_0^2(1+\mathrm{i}\phi)}{1+\phi^2}+\mathrm{i}\Omega\rho\right)F+\left(\frac{\sigma B_0^2(1+\mathrm{i}\phi)}{1+\phi^2}\right)\left(\frac{Q+\mathrm{i}P}{2h}\right)=0, \tag{20}$$

in which the boundary conditions are

$$F(h)=\Omega l+U_1+\mathrm{i}U_2,\quad F(-h)=-(\Omega l+U_1+\mathrm{i}U_2), \tag{21}$$

$$F=f+\mathrm{i}g,\quad \bar F=f-\mathrm{i}g. \tag{22}$$

At this stage it is convenient to define the dimensionless quantities as follows:

$$z^*=\frac{z}{h},\quad F^*=\frac{F}{\Omega l},\quad b_1^*=\frac{b}{a}\left(\frac{\Omega l}{h}\right)^{n-1},$$

$$M^2=\frac{\sigma B_0^2h^2}{a},\quad R=\Omega\rho\frac{h^2}{a},\quad V_1=\frac{U_1}{\Omega l},\quad V_2=\frac{U_2}{\Omega l}.$$

The resulting dimensionless problem reduces to

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2}+b_1\left[\frac{\mathrm{d}^2F}{\mathrm{d}z^2}\left(\frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}\bar F}{\mathrm{d}z}\right)^{\frac{n-1}{2}}+\frac{n-1}{2}\frac{\mathrm{d}F}{\mathrm{d}z}\left(\frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}\bar F}{\mathrm{d}z}\right)^{\frac{n-3}{2}}\left(\frac{\mathrm{d}^2F}{\mathrm{d}z^2}\frac{\mathrm{d}\bar F}{\mathrm{d}z}+\frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}^2\bar F}{\mathrm{d}z^2}\right)\right]-\left(\frac{M^2(1+\mathrm{i}\phi)}{1+\phi^2}+\mathrm{i}R\right)F=-\left(\frac{M^2(1+\mathrm{i}\phi)}{1+\phi^2}\right)\left(\frac{Q+\mathrm{i}P}{2h\Omega l}\right), \tag{23}$$

$$F(1)=1+V_1+\mathrm{i}V_2,\quad F(-1)=-(1+V_1+\mathrm{i}V_2), \tag{24}$$

where the asteriks have been suppressed for simplicity.

## 3. Analytic Solution by HAM

For series solutions of (23) and (24) we take

$$F_0(z)=(1+V_1+\mathrm{i}V_2)z, \tag{25}$$

$$\mathcal{L}_1(F)=F'', \tag{26}$$

$$\mathcal{L}_1[D_1+zD_2]=0 \tag{27}$$

as the initial guess $F_0$ and auxiliary linear operator $\mathcal{L}_1$, respectively, and $D_i$ $(i=1,2)$ are the arbitrary constants.

The zeroth-order deformation problem is written as

$$(1-p)\mathcal{L}_1[\hat F(z;p)-F_0(z)]=p\hbar\mathcal{N}_1[\hat F(z;p)], \tag{28}$$

$$\hat F(1;p)=1+V_1+\mathrm{i}V_2,\quad \hat F(-1;p)=-(1+V_1+\mathrm{i}V_2), \tag{29}$$

where $p\in[0,1]$ is an embedding parameter and $\hbar$ is the auxiliary parameter and the nonlinear operator $\mathcal{N}_1$ is

$$\mathcal{N}_1[\hat F(z;p)]=\frac{\partial^2\hat F(z;p)}{\partial z^2}+b_1\left[\frac{\partial^2\hat F}{\partial z^2}\left(\frac{\partial\hat F}{\partial z}\frac{\partial\hat{\bar F}}{\partial z}\right)^{\frac{n-1}{2}}+\frac{n-1}{2}\frac{\partial\hat F}{\partial z}\left(\frac{\partial\hat F}{\partial z}\frac{\partial\hat{\bar F}}{\partial z}\right)^{\frac{n-3}{2}}\left(\frac{\partial^2\hat F}{\partial z^2}\frac{\partial\hat{\bar F}}{\partial z}+\frac{\partial\hat F}{\partial z}\frac{\partial^2\hat{\bar F}}{\partial z^2}\right)\right]-\left(\frac{M^2(1+\mathrm{i}\phi)}{1+\phi^2}+\mathrm{i}R\right)\hat F+\frac{M^2(1+\mathrm{i}\phi)}{1+\phi^2}\frac{Q+\mathrm{i}P}{2h\Omega l}. \tag{30}$$

Obviously for $p=0$ and $p=1$ one has

$$\hat{F}(z,0)=F_0(z),\quad \hat{F}(z,1)=F(z). \tag{31}$$

When $p$ increases from 0 to 1, $\hat{F}(z,p)$ varies from $F_0(z)$ to the solution $F(z)$. By Taylors' series and (31), we get

$$\hat{F}(z;p)=F_0(z)+\sum_{m=1}^{\infty}F_m(z)p^m,\quad F_m(z)=\frac{1}{m!}\frac{\partial^m\hat{F}(z,p)}{\partial p^m}\Big|_{p=0}. \tag{32}$$

Choosing $\hbar_1$ properly so that the above series is convergent at $p=1$, one can write

$$\hat{F}(z)=F_0(z)+\sum_{m=1}^{\infty}F_m(z). \tag{33}$$

The $m$th-order deformation problem is

$$\mathcal{L}_1[F_m(z)-\chi_m F_{m-1}(z)]=\hbar_1\mathcal{R}_m(z), \tag{34}$$

$$F_m(1)=F_m(-1)=0, \tag{35}$$

$$\mathcal{R}_m(z)=F''_{m-1}(z)-\left(\frac{M^2(1+\mathrm{i}\phi)}{1+\phi^2}+\mathrm{i}R\right)F_{m-1}(z)+(1-\chi_m)\frac{M^2(1+\mathrm{i}\phi)}{1+\phi^2}\frac{Q+\mathrm{i}P}{2h\Omega l}+b_1\Theta(z), \tag{36}$$

$$\Theta(z)=\sum_{t=0}^{2m+1}(2\delta_{m,t}+\Delta_{m,t})z^t,\ \text{ for } n=3,\quad =\sum_{t=0}^{2m+3}(2\gamma_{m,t}+3\lambda_{m,t})z^t\ \text{ for } n=5, \tag{37}$$

$$\chi_m=\begin{cases}0, & m\leqslant 1,\\ 1, & m>1.\end{cases} \tag{38}$$

The $m$th-order deformation problem has been solved by using MATHEMATICA up to the first few orders of approximations for $n=3$ and $n=5$. The solution is expressed by

$$F_m(z)=\sum_{t=0}^{2m+n-2}C_{m,t}z^t,\quad C_{m,t}=a_{m,t}+\mathrm{i}b_{m,t},\quad m\geqslant 0, \tag{39}$$

in which for $m\geqslant 1$ and $0\leqslant t\leqslant 2m+n-2$, the recurrence formulas are

$$C_{m,0}=\chi_m\chi_{2m+1}C_{m-1,0}-\left(\frac{A}{2}-\sum_{t=0}^{m+\frac{1}{2}(n-1)}\frac{\Gamma_{m,2t}}{(1+2t)(2+2t)}\right),$$

$$C_{m,1}=\chi_m\chi_{2m}C_{m-1,1}-\Gamma_{m,1}\frac{1}{3},$$

$$C_{m,2}=\chi_m\chi_{2m-1}C_{m-1,2}+\frac{1}{2}(A+\Gamma_{m,0}),$$

$$C_{m,t}=\chi_m\chi_{2m-t+1}C_{m-1,t}+\Gamma_{m,t-2}\frac{1}{(t-1)t},\quad 3\leqslant t\leqslant 2m+1,$$

$$\Gamma_{m,t}=\hbar\left\{\chi_{2m-t+n-2}\left[e_{m-1,t}-\left(\frac{M^2(1+\mathrm{i}\phi)}{1+\phi^2}+\mathrm{i}R\right)C_{m-1,t}\right]+b_1\Pi(z)\right\},$$

$$\Pi(z)=2\delta_{m,t}+\Delta_{m,t}\ \text{ for } n=3,$$

$$\Pi(z)=3\lambda_{m,t}+2\gamma_{m,t}\ \text{ for } n=5,$$

$$A=\hbar(1-\chi_m)\frac{Q+\mathrm{i}P}{2h\Omega l},$$

and the related coefficients $\delta_{m,t}$, $\Delta_{m,t}$, $\lambda_{m,t}$, $\gamma_{m,t}$ for $m\geqslant 1, 0\leqslant t\leqslant 2m+n-2$ are

$$\delta_{m,t}=\sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{q=\max\{0,1+t-2k-2m\}}^{\min\{t,2k+2\}}d_{m-1-k,t-q}\cdot\sum_{u=\max\{0,q-1-2l\}}^{\min\{q,2k-2l+1\}}\hat{d}_{k-l,u}e_{l,q-u},$$

$$\Delta_{m,t}=\sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{u=\max\{0,1+t+2k-2m\}}^{\min\{t,2k+2\}}d_{m-1-k,t-u}\cdot\sum_{q=\max\{0,u+1+2l-2k\}}^{\min\{u,2l+1\}}d_{k-l,u-q}\hat{e}_{l,q},$$

$$\lambda_{m,t}=\sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{a=0}^{l}\sum_{b=0}^{a}\sum_{w=\max\{0,2k+1+t-2m\}}^{\min\{t,2k+4\}}d_{m-1-k,t-w}\cdot\sum_{u=\max\{0,2l+w-1-2k\}}^{\min\{w,2l+3\}}d_{k-l,w-u}\sum_{r=\max\{0,2a+u-2l-1\}}^{\min\{u,2a+2\}}\hat{d}_{l-a,u-r}\cdot\sum_{p=\max\{0,r-1-2b\}}^{\min\{r,2b+1\}}\hat{d}_{a-b,p}\hat{e}_{b,r-p},$$

$$\gamma_{m,t} = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{a=0}^{l}\sum_{b=0}^{a}\sum_{w=\max\{0,2k+2+t-2m\}}^{\min\{t,2k+4\}} d_{m-1-k,t-w} \cdot \sum_{u=\max\{0,2l+w-1-2k\}}^{\min\{w,2k-2l+1\}} d_{k-l,w-u} \sum_{r=\max\{0,2a+u-2l-1\}}^{\min\{u,2a+2\}} \hat{d}_{l-a,u-r} \cdot \sum_{p=\max\{0,r-1-2b\}}^{\min\{r,2b+1\}} \hat{d}_{a-b,p} e_{b,r-p},$$

$$C_{m,t} = a_{m,t} + \mathrm{i} b_{m,t}, \quad \hat{C}_{m,t} = a_{m,t} - \mathrm{i} b_{m,t},$$
$$d_{m,t} = (1+t)c_{m,1+t}, \quad \hat{d}_{m,t} = (1+t)\hat{c}_{m,1+t}, \tag{40}$$
$$e_{m,t} = (1+t)d_{m,1+t}, \quad \hat{e}_{m,t} = (1+t)\hat{d}_{m,1+t}.$$

The corresponding $m$th-order approximation is

$$\sum_{m=0}^{M} F_m(z) = \sum_{m=1}^{2M} C_{m,0} + \sum_{m=1}^{2M+n-2}\sum_{t=0}^{2m+n-3} C_{m,t} z^t, \tag{41}$$

and the explicit solution is

$$F(z) = \lim_{M\to\infty}\left[\sum_{m=0}^{M} F_m(z)\right] = \lim_{M\to\infty}\left[\sum_{m=1}^{2M} C_{m,0} + \sum_{m=1}^{2M+n-2}\sum_{t=0}^{2m+n-3} C_{m,t} z^t\right]. \tag{42}$$

## 4. Heat Transfer Analysis

This section deals with the heat transfer analysis. The corresponding temperatures of the lower ($z=-h$) and upper ($z=h$) disks are

$$\tau = \tau_1 \text{ at } z=-h, \quad \tau = \tau_2 \text{ at } z=h. \tag{43}$$

The dimensionless form of the energy equation is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}z^2} = -E_c P_r \left[1 + b_1 \left\{\left(\frac{\mathrm{d}f^*}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}g^*}{\mathrm{d}z}\right)^2\right\}^{\frac{n-1}{2}}\right] \cdot \left(\frac{\mathrm{d}f^*}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}g^*}{\mathrm{d}z}\right)^2, \tag{44}$$

where $f^* = f/\Omega l$, $g^* = g/\Omega l$, $b_1 = (\Omega l)^{n-1} b/a$, $C_p$ is the specific heat, $P_r = aC_p/K$ is the Prandtl number, $E_c = (\Omega l)^2/(\tau_1 - \tau_2)$ is the Eckert number and $\theta = \frac{\tau-\tau_2}{\tau_1-\tau_2}$. Here $\tau_1 > \tau_2$, so that $E_c > 0$.

In terms of $F$ (44) is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}z^2} = -B_r \left[1 + b_1\left(\frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}\bar{F}}{\mathrm{d}z}\right)^{\frac{n-1}{2}}\right] \frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}\bar{F}}{\mathrm{d}z}, \tag{45}$$

subject to the boundary conditions

$$\theta(-1) = 1, \quad \theta(1) = 0, \tag{46}$$

where $B_r = E_c P_r$ is the Brinkman number.

Selecting the initial guess $\theta_0$ and the auxiliary linear operator $\mathcal{L}_2$ of the form

$$\theta_0(z) = \frac{1}{2}(1-z), \tag{47}$$

$$\mathcal{L}_2(\hat{\theta}) = \theta'', \tag{48}$$

$$\mathcal{L}_2[B_1 + zB_2] = 0, \tag{49}$$

the zeroth-order problem becomes

$$(1-p)\mathcal{L}_2[\hat{\theta}(z;p) - \theta_0(z)] = p\hbar\mathcal{N}_2[\hat{F}(z;p), \hat{\theta}(z;p)], \tag{50}$$

$$\hat{\theta}(-1;p) = 1, \quad \hat{\theta}(1;p) = 0, \tag{51}$$

$$\mathcal{N}_2[\hat{F}(z;p)] = \frac{\partial^2\theta}{\partial z^2} + E_c P_r \left[1 + b_1\left(\frac{\partial\hat{F}}{\partial z}\frac{\partial\bar{\hat{F}}}{\partial z}\right)^{\frac{n-1}{2}}\right] \frac{\partial\hat{F}}{\partial z}\frac{\partial\bar{\hat{F}}}{\partial z}, \tag{52}$$

in which $B_i$ ($i=1,2$) are arbitrary constants. The problem at the $m$th order satisfies the following equations:

$$\mathcal{L}_2[\theta_m(z) - \chi_m\theta_{m-1}(z)] = \hbar_1 \mathcal{S}_m(z), \tag{53}$$

$$\theta_m(-1) = 1, \quad \theta_m(1) = 0, \tag{54}$$

$$\mathcal{S}_m(z) = \theta''_{m-1}(z) + E_c P_r \sum_{k=0}^{m-1} F'_{m-1-k}\bar{F}'_k + b_1\Psi(z), \tag{55}$$

$$\Psi(z) = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{a=0}^{l}[F'_{m-1-k}F'_{k-l}\bar{F}'_{l-a}\bar{F}'_a], \text{ for } n=3,$$
$$= \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{a=0}^{l}\sum_{b=0}^{a}\sum_{c=0}^{b}[F'_{m-1-k}F'_{k-l}F'_{1-a}\bar{F}'_{a-b}\bar{F}'_{b-c}\bar{F}'_c] \text{ for } n=5. \tag{56}$$

When $n=3$ and $n=5$ the solution is

$$\theta_m(z) = \sum_{t=0}^{2m+n-3} A_{m,t} z^t, \quad m \geqslant 0, \tag{57}$$

in which for$m \geqslant 1$ and $0 \leqslant t \leqslant 2m+n-2$, we have

$$A3_{m,0} = \chi_m\chi_{2m}A3_{m-1,0} - \sum_{t=0}^{2m+n-3}\frac{\zeta_{m,2t}}{(2t+1)(2t+2)},$$

$$A3_{m,1} = \chi_m\chi_{2m-1}A3_{m-1,1} - \sum_{t=0}^{2m+n-3}\frac{\zeta_{m,2t+1}}{(2t+2)(2t+2)},$$

$$A3_{m,2} = \chi_m\chi_{2m-2}A3_{m-1,2} + \frac{1}{2}\zeta_{m,0},$$

$$A3_{m,t} = \chi_m\chi_{2m-t}A3_{m-1,t} + \zeta_{m,t-2}\frac{1}{(t-1)t},$$
$$3 \leqslant t \leqslant 2m+1,$$

$$\zeta_{m,t} = \hbar\sum_{t=0}^{2m+n-1}[\chi_{2m+n-1}(A3_{m,t} + E_cP_r\alpha_{m,t}) + b_1E_cP_r\Pi(z)],$$

$$\Pi(z) = \sum_{t=0}^{2m+2}\beta_{m,t} \text{ for } n = 3, \tag{58}$$

$$\Pi(z) = \sum_{t=0}^{2m+4}\gamma_{m,t} \text{ for } n = 5, \tag{59}$$

and the coefficients $\alpha_{m,t}$, $\beta_{m,t}$, $\gamma_{m,t}$ for $m \geqslant 1, 0 \leqslant t \leqslant 2m+n-3$ are

$$\alpha_{m,t} = \sum_{k=0}^{m-1}\sum_{q=\max\{0,t-1-2k\}}^{\min\{t,2m-2k-1\}} d_{m-1-k,q}\hat{d}_{k,t-q},$$

$$\beta_{m,t} = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{a=0}^{l}\sum_{u=\max\{0,1+t+2k-2m\}}^{\min\{t,2k+2\}} d_{m-1-k,t-u} \cdot \sum_{r=\max\{0,u+2l-2k\}}^{\min\{u,2l+2\}} d_{k-l,u-r} \sum_{p=\max\{0,r-2a-1\}}^{\min\{r,2l-2a+1\}} \hat{d}_{a,r-p}\hat{d}_{l-a,p},$$

$$\gamma_{m,t} = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{r=0}^{l}\sum_{s=0}^{r}\sum_{a=0}^{s}\sum_{v=\max\{0,2k+1+t-2m\}}^{\min\{t,2k+5\}} d_{m-1-k,t-v} \cdot \sum_{w=\max\{0,2l+v-1-2k\}}^{\min\{v,2l+4\}} d_{k-l,t-w} \sum_{u_1=\max\{0,2r+w-2l-1\}}^{\min\{w,2r+3\}} d_{l-r,w-u_1} \cdot \sum_{p_1=\max\{0,2s+u_1-2r-1\}}^{\min\{u_1,2s+2\}} \hat{d}_{r-s,u_1-p_1} \cdot \sum_{q=\max\{0,2a+p_1-2s-1\}}^{\min\{p_1,2a+1\}} \hat{d}_{s-a,p_1-q}\hat{d}_{a,q},$$

$$A2_{m,p} = (1+p)A_{m,p+1},\ A3_{m,p} = (1+p)A2_{m,p+1}. \tag{60}$$

The corresponding $m$th-order approximation is

$$\sum_{m=0}^{M}\theta_m(z) = \sum_{m=1}^{2M}A3_{m,0} + \sum_{m=1}^{2M+n-3}\sum_{t=0}^{2m+n-4}A3_{m,t}z^t \tag{61}$$

and the explicit expression for the series solution is

$$\theta(z) = \lim_{M\to\infty}\left[\sum_{m=0}^{M}\theta_m(z)\right] = \lim_{M\to\infty}\left[\sum_{m=1}^{2M}A3_{m,0} + \sum_{m=1}^{2M+n-3}\sum_{t=0}^{2m+n-4}A3_{m,t}z^t\right]. \tag{62}$$

## 5. Convergence of the Solution

As pointed out by Liao [32], the convergence of the series solutions (42) and (62) strongly depend on the values of $\hbar_1$, $\hbar_2$, and $\hbar$. For this purpose $\hbar_1$, $\hbar_2$, and $\hbar$-curves are drawn in Figures 1 – 3. It is noted that admissible values of $\hbar_1$ and $\hbar_2$ are $-0.45 \le \hbar_1 \le 0.05$, $-0.5 \le \hbar_2 \le -0.05$ when $n = 3$ and $-0.4 \le \hbar_1 \le -0.1$, $-0.45 \le \hbar_2 \le -0.1$ when $n = 5$. For the temperature $-2.5 \le \hbar \le 0.1$ when $n = 3$ and $-2.5 \le \hbar \le -0.2$ when $n = 5$. Our calculations indicate that the series given by (42) and (62) converge in the whole region of $z$ when $\hbar_1 = -0.25$, $\hbar_2 = -0.32$, and $\hbar = -1$ for $n = 3$, whereas $\hbar_1 = -0.27$, $\hbar_2 = -0.32$, and $\hbar = -1$ for $n = 5$.

## 6. Results and Discussion

Figures 4 – 21 show the profiles of $f/\Omega l$, $g/\Omega l$, and the temperature $\theta$ for various values of $b$ and $M$ with and without Hall effect for $n = 3$ and $n = 5$ in viscous and Sisko fluids. From Figures 4 and 5 we observe that the velocity profiles decrease by increasing $b$ for $n = 3$ and $\phi = 0$. Moreover, the same trend of velocity profiles is noted for $\phi \neq 0$. However, the behaviour of $b$ on the velocity profiles for $n = 5$ is quite opposite in Figures 10 and 11. The variations of $M$ on the velocity profiles for $n = 3$, $\phi = 0$, and $\phi \neq 0$ are for viscous and Sisko fluids shown in Figures 6 – 9. In these Figures it can be seen that velocity profiles are greater for viscous fluid in comparison to the Sisko fluid. Further, the velocities increase for large values of $M$ for $\phi = 0$ and $\phi \neq 0$ in both fluids. Figures 12 – 15 are prepared for the variation of $M$ on the velocity profiles for $n = 5$. Here the variation in the velocity profiles is not much greater for the viscous fluid for compared with that of the Sisko fluid. The variation of $b$ and Brinkman number $B_r$ on the temperature for $\phi = 0$ and $\phi \neq 0$ is shown in the Figures 16 – 21. Here it is evident that the temperature increases for increasing $b$ in both cases for $\phi = 0$ and $\phi \neq 0$. Furthermore, the variation of $B_r$ with the temperature is qualitatively similar to that of $b$ in both cases $\phi = 0$ and $\phi \neq 0$ for both fluids. However,

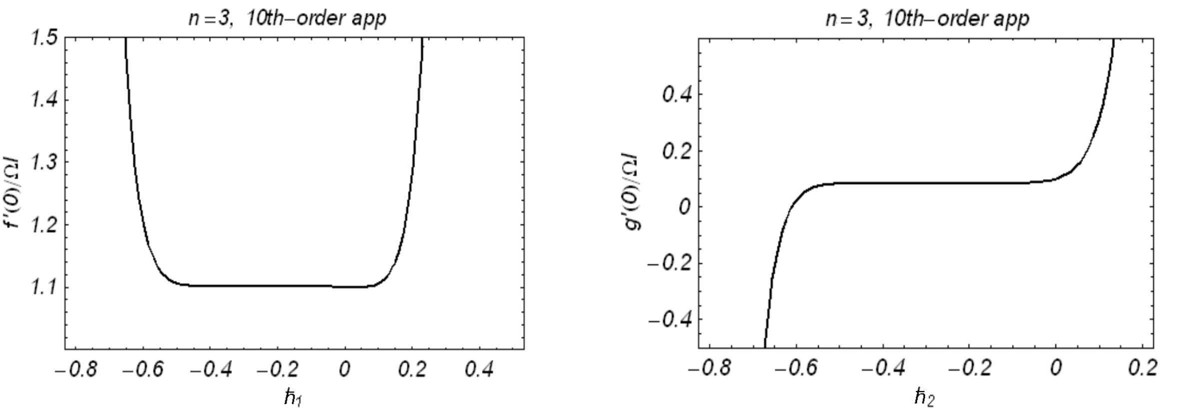


Fig. 1. $\hbar_1$ and $\hbar_2$-curves of the velocity for the 10th-order approximation for $n = 3$.

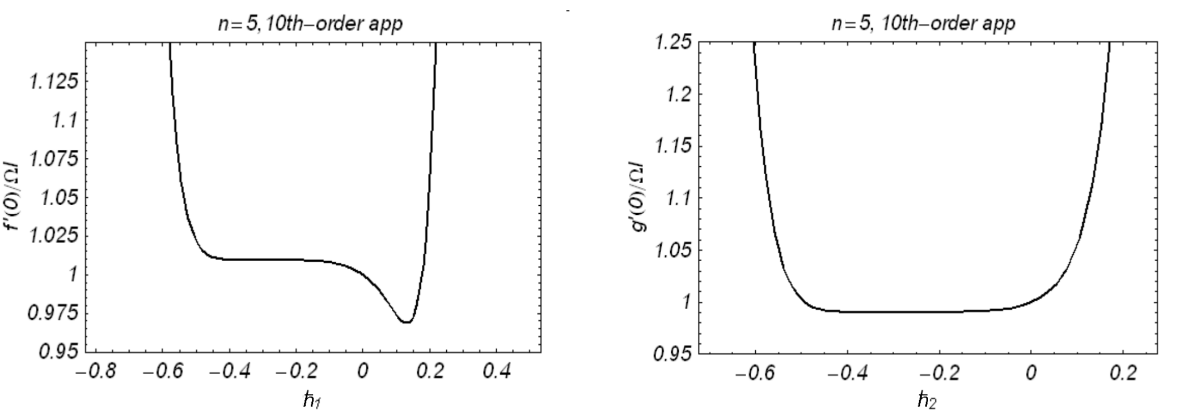


Fig. 2. $\hbar_1$ and $\hbar_2$-curves of the velocity for the 10th-order approximation for $n = 5$.

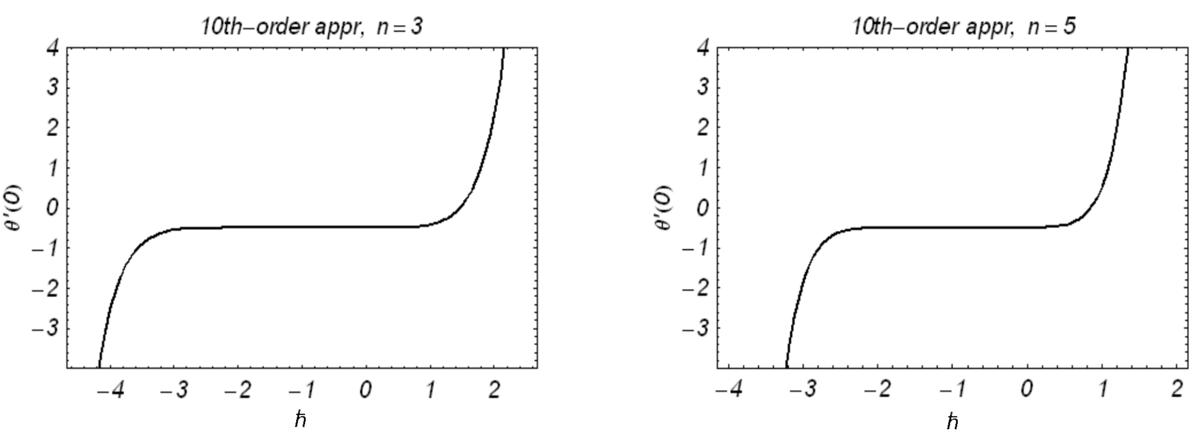


Fig. 3. $\hbar$-curve of the temperature for the 10th-order approximation for $n = 3$ and $n = 5$.

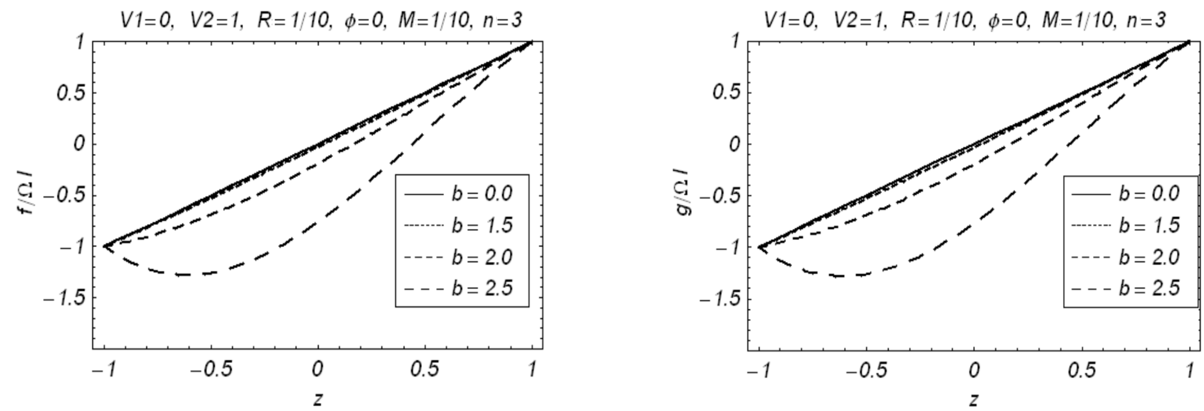

Fig. 4. Velocity profile for different values of $b$ for $M = 1/10$, $V_1 = 0$, $V_2 = 0$, $\phi = 0$, and $n = 3$.

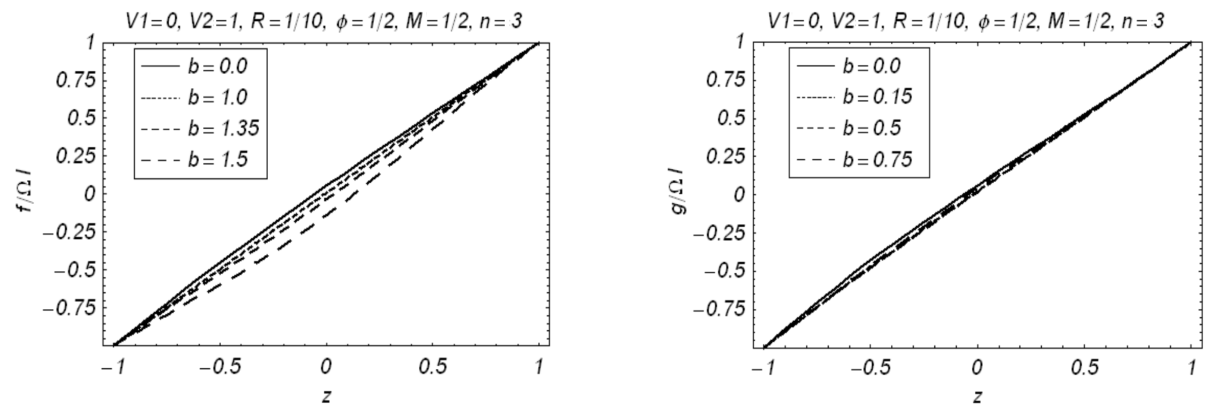

Fig. 5. Velocity profile for different values of $b$ for $M = 1/10$, $V_1 = 0$, $V_2 = 0$, $\phi = 1/2$, and $n = 3$.

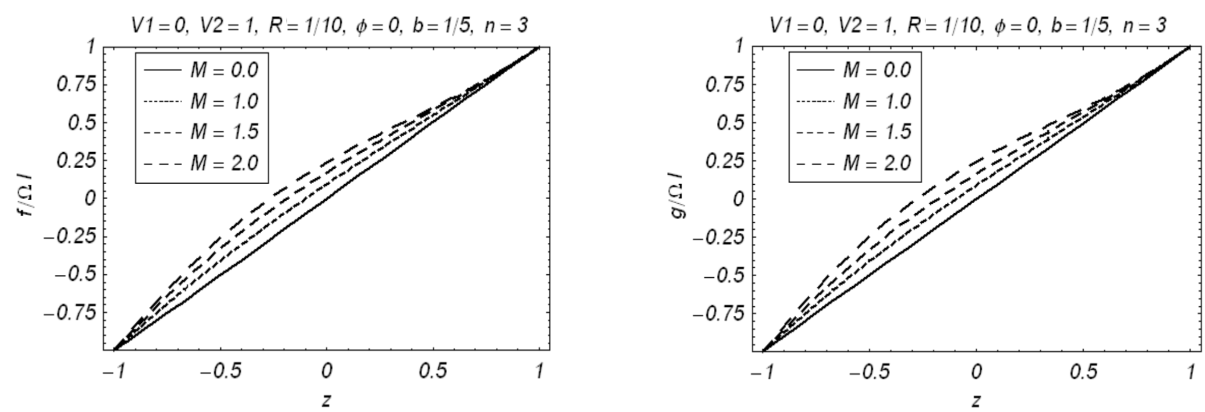

Fig. 6. Velocity profile for different values of $M$ for $b = 1/5$, $V_1 = 0$, $V_2 = 0$, $\phi = 0$, and $n = 3$.

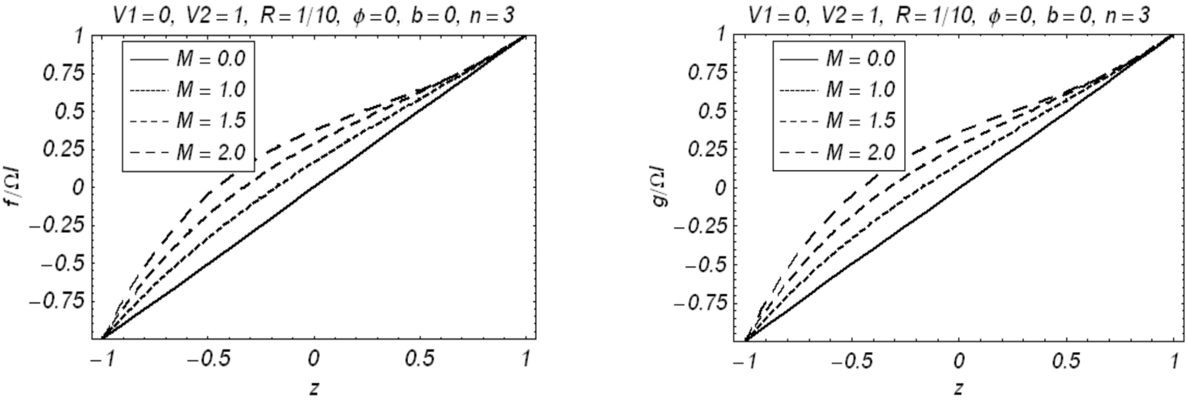


Fig. 7. Velocity profile for different values of $M$ for $b = 0$, $V_1 = 0$, $V_2 = 0$, $\phi = 0$, and $n = 3$.

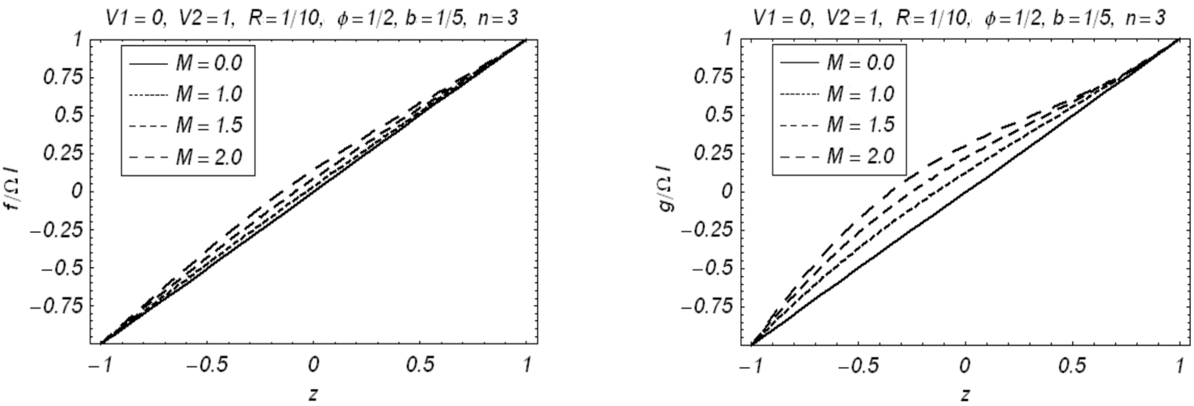


Fig. 8. Velocity profile for different values of $M$ for $b = 1/5$, $V_1 = 0$, $V_2 = 0$, $\phi = 1/2$, and $n = 3$.

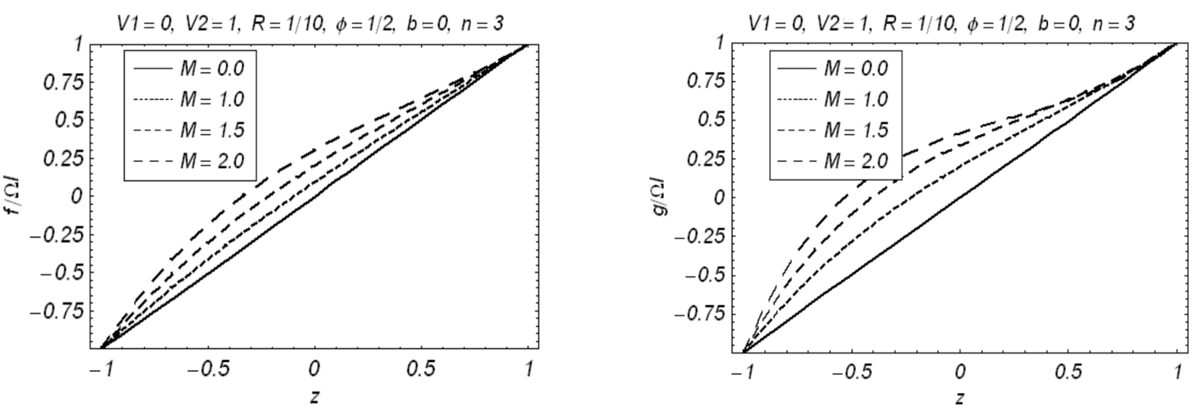


Fig. 9. Velocity profile for different values of $M$ for $b = 0$, $V_1 = 0$, $V_2 = 0$, $\phi = 1/2$, and $n = 3$.

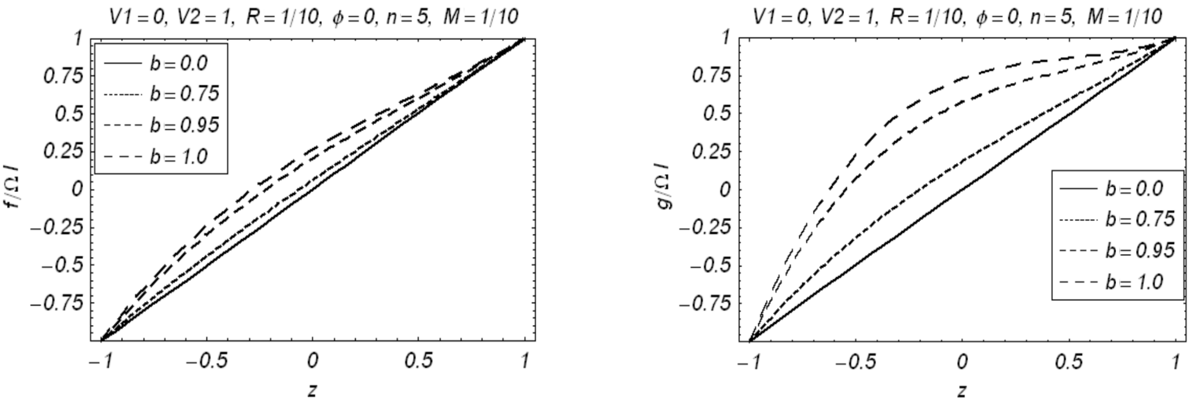


Fig. 10. Velocity profile for different values of *b* for $M = 1/10$, $V_1 = 0$, $V_2 = 0$, $\phi = 0$, and $n = 5$.

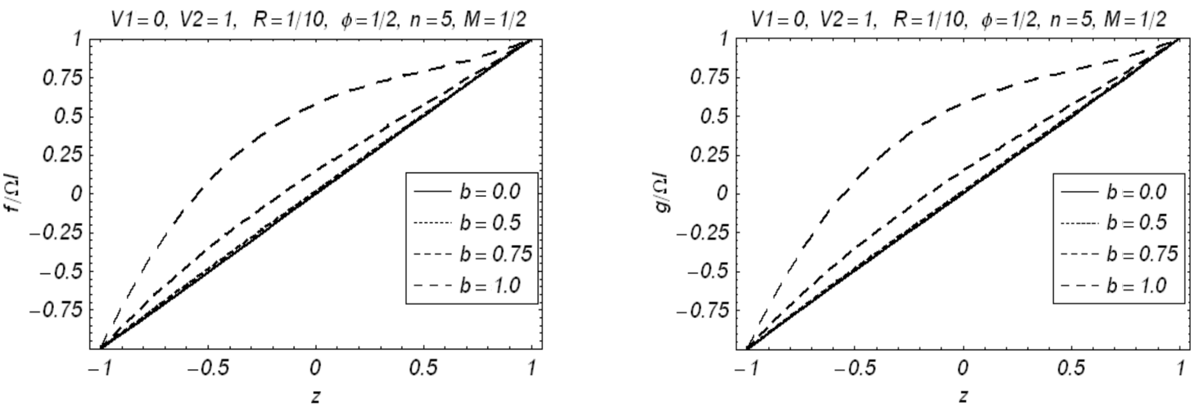


Fig. 11. Velocity profile for different values of *b* for $M = 1/10$, $V_1 = 0$, $V_2 = 0$, $\phi = 1/2$, and $n = 5$.

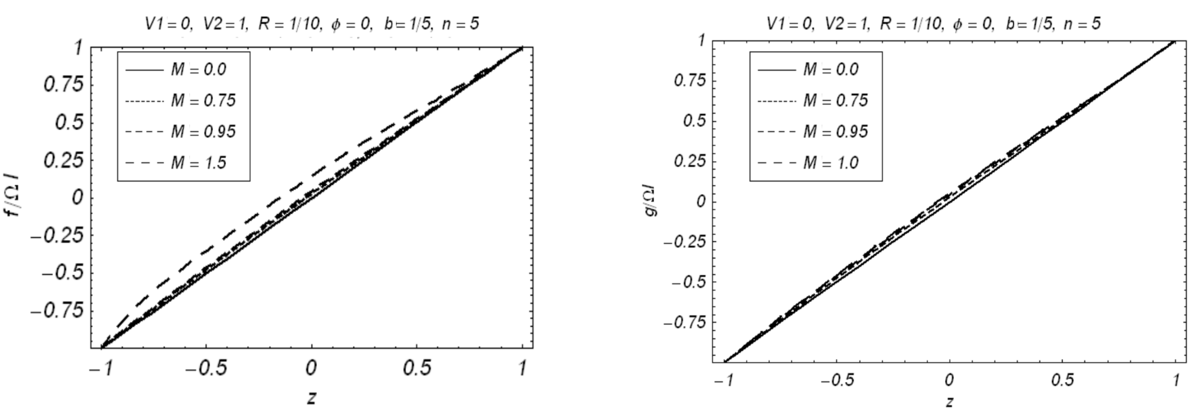


Fig. 12. Velocity profile for different values of *M* for $b = 1/5$, $V_1 = 0$, $V_2 = 0$, $\phi = 0$, and $n = 5$.

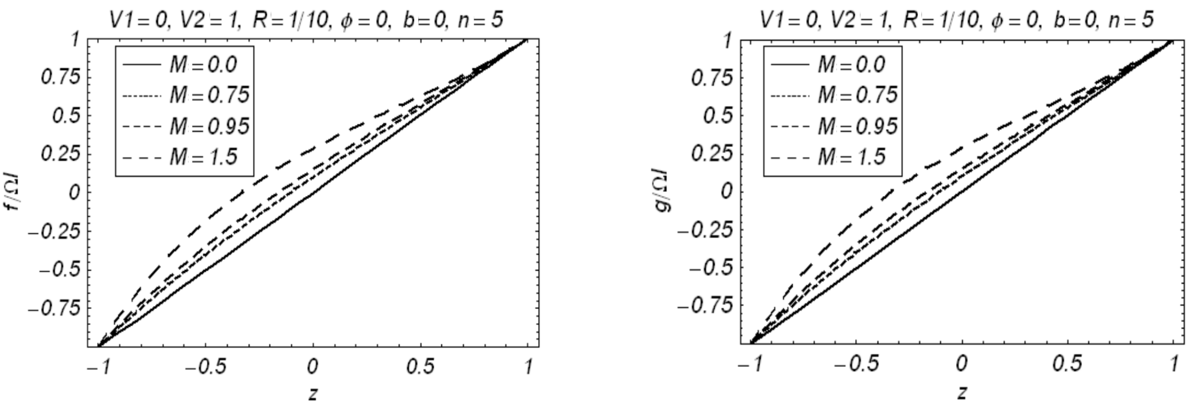


Fig. 13. Velocity profile for different values of $M$ for $b = 0$, $V_1 = 0$, $V_2 = 0$, $\phi = 0$, and $n = 5$.

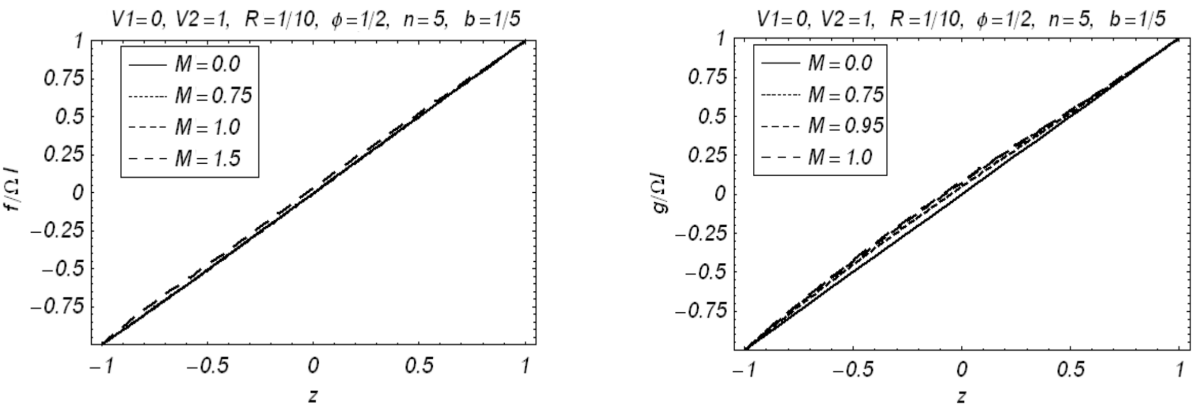


Fig. 14. Velocity profile for different values of $M$ for $b = 1/5$, $V_1 = 0$, $V_2 = 0$, $\phi = 1/2$, and $n = 5$.

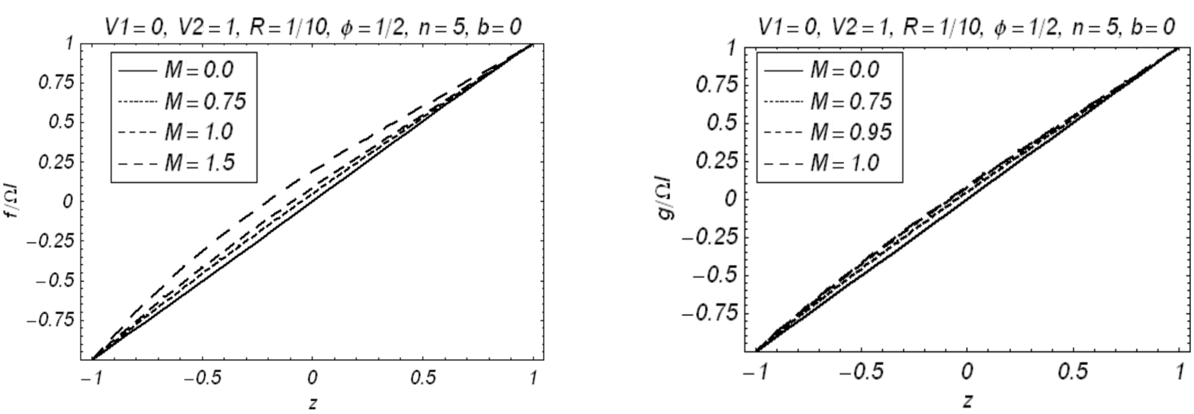


Fig. 15. Velocity profile for different values of $M$ for $b = 0$, $V_1 = 0$, $V_2 = 0$, $\phi = 1/2$, and $n = 5$.

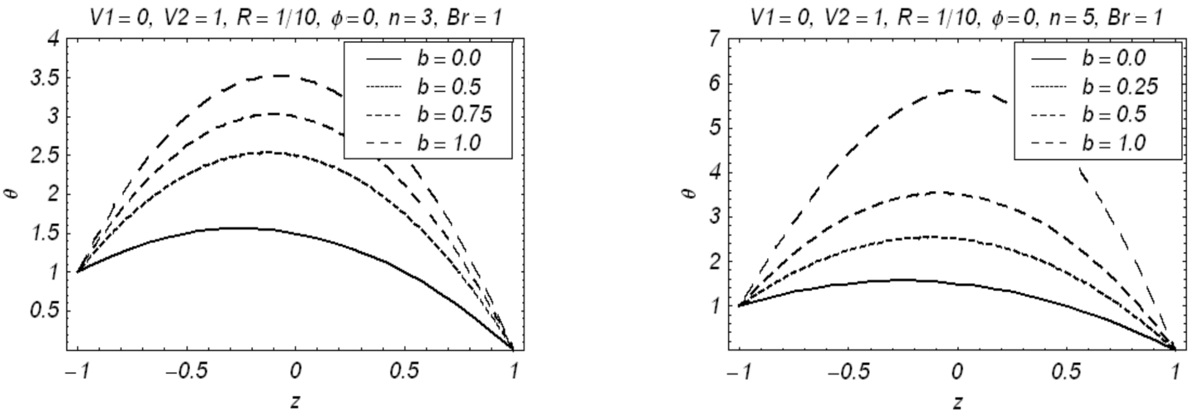


Fig. 16. Temperature for different values of $b$ for $Br = 1$, $V_1 = 0$, $V_2 = 0$, $\phi = 0$, and $R = 1/10$.

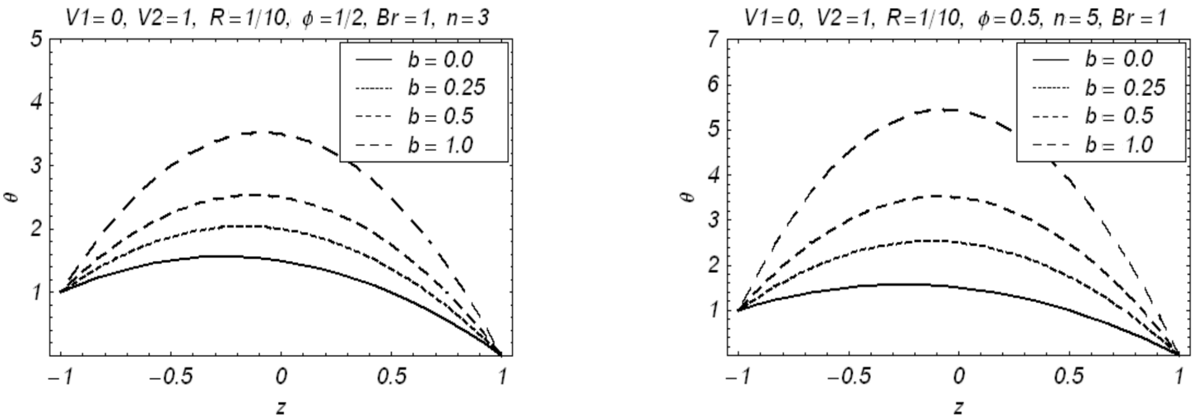


Fig. 17. Temperature for different values of $b$ for $Br = 1$, $V_1 = 0$, $V_2 = 0$, $\phi = 1/2$, and $R = 1/10$.

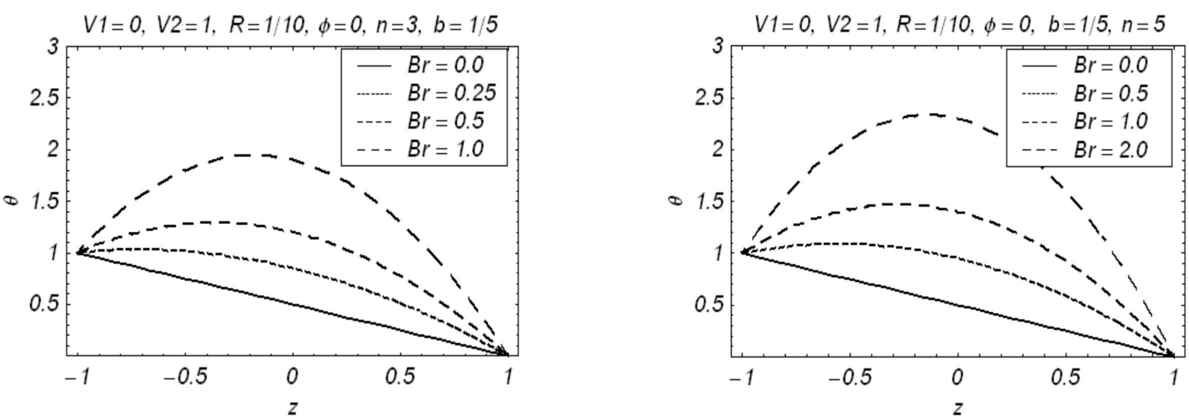


Fig. 18. Temperature for different values of $Br$ for $b = 1/5$, $V_1 = 0$, $V_2 = 0$, $\phi = 0$, and $R = 1/10$.

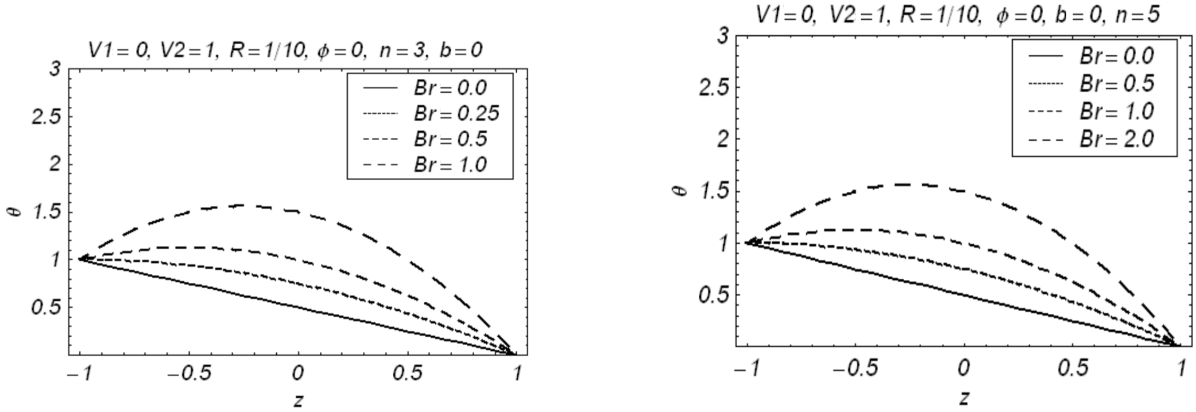


Fig. 19. Temperature for different values of *Br* for $b = 0$, $V_1 = 0$, $V_2 = 0$, $\phi = 0$, and $R = 1/10$.

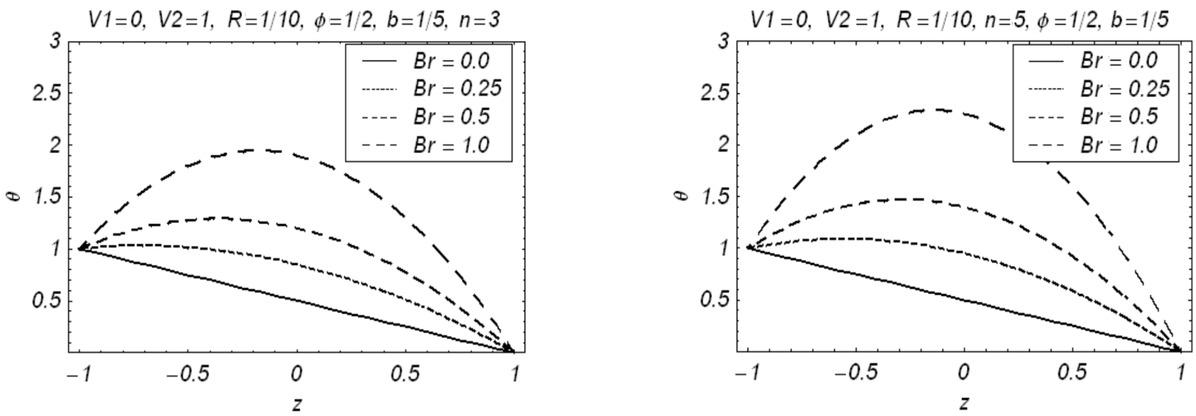


Fig. 20. Temperature for different values of *Br* for $b = 1/5$, $V_1 = 0$, $V_2 = 0$, $\phi = 1/2$, and $R = 1/10$.

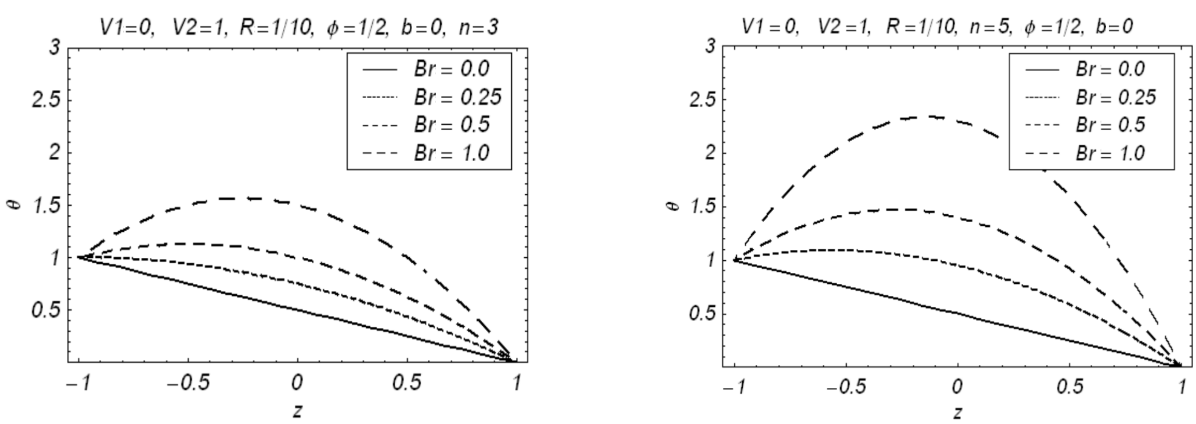


Fig. 21. Temperature for different values of *Br* for $b = 0$, $V_1 = 0$, $V_2 = 0$, $\phi = 1/2$, and $R = 1/10$.

it is noted that the temperature for $n = 5$ is larger compared with that of $n = 3$ for viscous and Sisko fluids.

## 7. Concluding Remarks

Here the flow analysis of a Sisko fluid is discussed in the presence of Hall current. The heat transfer analysis is further analyzed. The flow problem for velocity and temperature distributions are first modeled and then solved analytically. Analytic solutions of the arising problems are developed by homotopy analysis method. To the best of our information the present problem is not studied yet. Even such analysis is not available in the literature with Hall and heat transfer effects.

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